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Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

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EXTENSIONS OF RESULTS OF PARDOUX ON STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS OF FILTERING

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ABSTRACT

Pardoux's results on the Zakai equation for nonlinear filtering are extended to cover the case of estimating a signal modified by a potential term. This is applied to state a rigorous Zakai equation for certain filtering problems involving signals with entrance boundaries.

AMS (MOS) Subject Classifications: 60H15, 62M05, 35R60, 93E10

Key Words: Zakai's equation, nonlinear filtering, entrance boundaries.

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SIGNIFICANCE AND EXPLANATION

A common problem in the analysis of stochastic systems is the estimation of the system's state given only noise-corrupted or incomplete observations. For instance, examples occur in communications theory when one wants to estimate a signal sent over a noisy channel. The problem of filtering is to build an estimate, i.e. filter, that provides the best information about the state given the observations.

Let $x(t)$ denote the state of the system. In the most common model, $x(t)$ is a Markov process modelling a differential equation driven by stochastic inputs, and it is observed via

$$y(t) = \int_0^t h(x(s))ds + W(t) \quad ,$$

where $W(t)$ is a Brownian motion. Since $W(t)$ has independent increments it is a good model for noise. The density of $x(t)$ given that $y(s)$, $0 < s < t$, is known, contains all information about $x(t)$ that is in $y(s)$, $0 < s < t$. Zakai and Pardoux have established partial differential equations for such conditional densities in the effort to compute them. This paper extends their results to a class of Markov signals evolving on bounded domains with entrance boundaries. This means that the process can enter its domain from the boundary, but cannot return to the boundary once inside. A typical example is the Bessel process; this is the process $r_t = |B_t|$, where B_t is a Brownian motion in 3-dimensional Euclidean space, and $|B_t|$ denotes its distance from the origin.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

EXTENSIONS OF RESULTS OF PARDOUX ON STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS OF FILTERING

Daniel Ocone

1. Introduction

Define a nonlinear filtering problem on the probability space (Ω, F, P) by the model

$$\begin{aligned} dx_t &= m(t,x_t)dt + \sigma(t,x_t)db_t & x_0 &= \eta \\ dy_t &= h(t,x_t)dt + dW_t & y_0 &= 0 \end{aligned} \quad . \tag{1}$$

Here, the signal x_t is taken to be $\mathbb{R}^N$-valued, and the observation $\mathbb{R}^P$-valued. As usual, $b(\cdot)$ and $W(\cdot)$ are independent Brownian motions, and η is a r.v. independent of them. Let $\tilde{P}$ be the measure on Ω defined in 2.1, and let $p(x,t)$ denote the density of x_t. If $F_t := \sigma\{y(s) | 0 < s < t\}$ then

$$\tilde{u}(t,x) = \tilde{E}\Big[\frac{dP}{d\tilde{P}} \;\Big|\; F_t, \; x_t = x\Big] p(x,t)$$

is an unnormalized conditional density of x_t given $y(s)$, $s < t$; that is,

$$\pi_t(f) := E\{f(x_t) | F_t\} = \frac{\int f(x)\tilde{u}(t,x)dx}{\int \tilde{u}(t,x)dx} \tag{2}$$

for all f s.t. $Ef^2(x_t) < \infty$.

One line of investigation in nonlinear filtering theory seeks to characterize $\tilde{u}(t,x)$ as the solution to a stochastic partial differential equation, the Zakai equation. Pardoux [7,8,9] has recently brought this approach to its most complete and rigorous form. After interpreting the formally derived Zakai equation variationally, he shows under mild assumptions that it has a unique solution which is indeed an unnormalized conditional

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density. Using the same methods, he also derives Zakai equations for signals diffusions evolving in bounded domains and having absorbing, elastic, or inelastic behaviors at the boundaries.

This note concerns itself with some variants of the basic filtering problem from the perspective of Zakai's equation. For example, rather than study estimates $\pi_t(f) = E\{f(x_t)|F_t\}$ one can consider

$$\pi_t^V(f) = E\{f(x_t)\ \exp[-\int_0^t V(s,x_s)ds]|F_t\} \quad ,$$

that is, estimates 'killed' by some potential V, and then look for corresponding 'killed' conditional densities $u(t,x)$ satisfying

$$\int f(x)u(x,t)dx = \sigma_t^V(f) = \tilde{E}[f(x_t)\ \exp[-\int_0^t V(s,x_s)ds]\frac{dP}{d\tilde{P}}\ |F_t] \quad . \tag{3}$$

In section 2, it is shown that this is easily accomplished using the techniques of Pardoux, and $\tilde{u}(t,x)$ is obtained under mild hypothesis as the solution of a Zakai equation in which V appears as a potential in the nonrandom operator term. In one direction, this yields a simple extension of Pardoux's generalized Feynman-Kac formula. A second variant is the filtering of 1-dimensional diffusions with entrance boundaries. Section 3 applies the 'killed' conditional density equations to derive Zakai equations for this class of signals. This work then rigorously justifies Zakai equations studied, but only formally derived, in Ocone [5,6].

2. Zakai equation extensions

2.1. Preliminaries

Our theory requires the following assumptions on the functions appearing in (1) and (3):

A_1) $V(t,x)$, $m_i(t,x)$, $i = 1 \cdots N$, $h_k(t,x)$ $k = 1,\cdots,p$ are bounded Borel functions

A_2) $\sigma_{ij}(t,x)$ are continuous functions satisfying $\frac{\partial \sigma_{ij}}{\partial x_i} \in L^\infty([0,T] \times R^N)$
$1 \le i,j \le N$

A_3) $A(t,x) = \sigma\sigma^*(t,x)$ is uniformly positive definite.

The conditions in $A_1) - A_3)$ are assumed to hold in the domain $[0,T] \times R^N$. The reasons for these assumptions will not be totally clear from the sketchy proofs to follow, but they are needed to make the details work.

Of course, given only these constraints, strong solutions of (1) will not in general exist. The martingale problem associated to (1), however, will have a nice solution. Let

$$L_t = \tfrac{1}{2} \sum_{i,j=1}^{N} A_{ij}(t,x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(t,x) \frac{\partial}{\partial x}$$

$$\Omega' = C([0,T]; R^N) \quad x(t)(w') = w'(t),\ w' \in \Omega''$$

$$\Omega'' = C([0,T]; R^P) \quad W(t)(w'') = w''(t),\ w'' \in \Omega''$$

$$\Omega = \Omega' \times \Omega''$$

$$\mu = \text{Wiener measure on } \Omega'' .$$

By the theory of Stroock and Varadhan [10] there exist measures Q_{sx} on Ω' solving the martingale problem with respect to L_t. Suppose η has a density $p_0(x)$ and set $Q_0(\cdot) = \int dx p_0(x) Q_{0x}(\cdot)$. As the solution of (1), we take

$$\begin{aligned} x(t)(w,w') &= x(t)(w') \\ y(t)(w,w') &= \int_0^t x(t)(w,w')dt - W(t)(w'') \end{aligned} \tag{4}$$

defined on $(\Omega,\ P = Q_0 \times \mu)$. We will need the following σ-algebras associated to (4),

$$F_t := \sigma\{y_i(s) \mid 0 < s < t,\ 1 < i < p\}$$

$$G_t := \sigma\{x_i(s),\ W_k(s) \mid 0 < s < t,\ 1 < i < N,\ 1 < k < p\} \quad .$$

To set up the unnormalized density, we need a new measure $\tilde{P}$ defined on Ω by the Girsanov formula

$$\frac{d\tilde{P}}{dP} = \exp\Big[-\int_0^t \langle h(x_s), dW_s\rangle - \tfrac{1}{2}\int_0^t \|h(x_s)\|^2 ds\Big] \quad .$$

It is well known that on $(\Omega, \tilde{P})$, $y(\cdot)$ is Brownian and independent of $x(\cdot)$, and

$$\pi_t^V(f) = \frac{\tilde{E}\big[f(x_t)\exp\big(-\int_0^t V(s,x_s)ds\big)Z_t \mid F_t\big]}{\tilde{E}\{Z_t \mid \ _t\}} \tag{5}$$

$$Z_t = \exp \int_0^t \langle h(x_s), dy_s\rangle - \tfrac{1}{2}\int_0^t \|h(x_s)\|^2 ds \quad .$$

Let $\sigma_t^V(f)$ denote the numerator of (5). Our goal is to find a representation $\sigma_t^V(f) = \int dx\, u(t,x)f(x)$ for some function space valued process $u(t,x,w)$.

2.2. The Zakai equation

If $x(t)$ has density $p(t,x)$, $u(t,x)$ should have the form

$$u(t,x) = \tilde{E}\Big[\exp\Big(-\int_0^t V(s,x_s)ds\Big)Z_t \mid F_t,\ x_t = x\Big]p(t,x)$$

and a formal analysis suggests that

$$\begin{aligned} du &= \Big[\tfrac{1}{2}\sum_{i,j=1}^{N} \frac{\partial^2}{\partial x_i \partial x_j} A_{ij}u - \sum_{i=1}^{N} \frac{\partial}{\partial x_i} b_i u - V(t,x)u\Big]dt \\ &\qquad + u(t,x)\,\langle h(t,x), dy_t\rangle \\ u(0,x) &= p_0(x) \quad . \end{aligned} \tag{6}$$

To work with (6) effectively, we adopt the variational interpretation of Pardoux [7]. Embed the Sobolev space $H^1(\mathbb{R}^N)$ in $L^2(\mathbb{R}^N)$, so that $H^1(\mathbb{R}^N) \subset L^2(\mathbb{R}^N) \subset H^{-1}(\mathbb{R}^N)$, and let $\|\cdot\|$, $|\cdot|$ be the norms in $H^1(\mathbb{R}^N)$, $L_2(\mathbb{R}^N)$ respectively, $\langle\cdot,\cdot\rangle$ the pairing between $H^1(\mathbb{R}^N)$ and $H^{-1}(\mathbb{R}^N)$. Recall that for $\varphi, \psi \in L^2(\mathbb{R}^N)$, $\langle\varphi,\psi\rangle$ equals the usual inner product. We now interpret the deterministic operator in (6) as the bounded operator $A_t : H^1(\mathbb{R}^N) \to H^{-1}(\mathbb{R}^N)$ defined by

$$\langle A_t u,v\rangle = -\int_{\mathbb{R}^N} \tfrac{1}{2} \sum_{i,j=1}^{N} A_{ij}(t,x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}\, dx$$
$$+ \int_{\mathbb{R}^N} \sum_{i=1}^{N} (b_i - \sum_{j=1}^{N} \frac{\partial A_{ij}}{\partial x_j}] \frac{\partial u}{\partial x_i}\, v\, dx - \int_{\mathbb{R}^N} V(t,x)u(x)v(t,x)dx$$
$$\forall u, v \in H^1(\mathbb{R}^N) \quad .$$

The adjoint $A_t^* : H^1(\mathbb{R}^N) \to H^{-1}(\mathbb{R}^N)$ may also be defined by this formula. Zakai's equation, in variational form, is then written

$$\begin{aligned} du(t) &= (A_t^* u(t))dt + u(t) \langle h(t,x),dy_t\rangle \\ u(0) &= p_0 \\ u(t) &\in L^2(\Omega \times [0,T]; H^1(\mathbb{R}^N)) \quad . \end{aligned} \tag{7}$$

2.3. The main theorems

In this section we discuss (7) and show that its solution is indeed the desired 'killed' conditional density. The proofs require only minor adaptions of techniques of Pardoux, and so, if discussed at all, will only be sketched. Two approaches to the theorems are possible and we will state results from both. However we will indicate proofs only for the analytically simpler method, so that readers, should they wish to verify any details, might have an easier time of it.

It is easy to show, using $A_1) - A_3)$, that A_t and A_t^* are coercive operators, uniformly in t. That is, there exist $\alpha > 0$ and λ such that

$$2\langle A_t u,u\rangle + \lambda|u|^2 \geq \alpha\|u\|^2 + \sum_{i=1}^{p} |h_i u|^2$$

$$\forall t \in [0,T] \quad \forall u \in H^1(R^N)$$

(and similarly for A_t^* since $\langle A_t^* u,u\rangle = \langle A_t u,u\rangle$). Coercivity is the basic fact underlying the theory.

<u>Theorem 1</u>. Suppose $p_0(x) \in L^1(R^N) \cap L^2(R^N)$ and $p_0(x) \geq 0$ a.e.

Then (7) possesses a unique solution $u(t)$, and moreover

i) $u(t) \in L^2(\Omega; C(0,T; L^2(R^N))) \cap L^1(\Omega; L^\infty(0,T; L^1(R^N)))$.

ii) Almost surely, $u(t,x) \geq 0$ a.e. $\forall t$.

<u>Proof</u>. Existence and uniqueness is a direct consequence of a general theorem of Pardoux [7]. That $u(t) \in L^1(\Omega; L^\infty(0,T; L^1(R^N)))$ and $u(t,x) \geq 0$ are proved by Pardoux [7] when $V = 0$, but the proofs depend only on the coercivity of A_t and extend to the present case.

Define $\tilde{\sigma}_t^V(f) = \int_{R^N} u(t,x)f(x)dx$. By theorem 1, $\tilde{\sigma}_t^V(f)$ can be thought of as a measurable process taking values in the space of bounded, positive measures on R^N.

<u>Theorem 2</u>. For $p_0(x) \in L^2(R^N) \cap L^1(R^N)$, $\tilde{\sigma}_t^V(f) = \sigma_t^V(f)$ a.e. $\forall t$, $\forall f \in L^\infty(R^N)$.

<u>Sketch of proof</u> (method of [7]). Let $\bar{\Pi}_{st}$ denote the two parameter semigroup

$$\bar{\Pi}_{st}f(x) + E_{sx}f(x_t)\exp -\int_s^t V(u,x_u)du \ .$$

We use $\bar{\Pi}_{st}$ to characterize σ_t^V as the solution of a certain equation.

<u>Lemma 1</u>. For every $f \in L^\infty(R^N)$

$$\sigma_t^V(f) = E\bar{\Pi}_{0t}f + \int_0^t \langle\sigma_s^V(h\bar{\Pi}_{st}f), dy_s\rangle \ . \tag{8}$$

Furthermore if σ_t is another process with values in the space of positive bounded measures and

$$\sigma_t(f) = E\bar{\Pi}_{0t}f + \int_0^t \langle\sigma_s(h\bar{\Pi}_{st}f), dy_s\rangle$$

then $\sigma_t(f) = \sigma_t^V(f)$ a.s. $\forall t$, $\forall f \in L^\infty(R^N)$.

<u>Proof</u>. The uniqueness part is a simple Gronwall-Bellman inequality argument (see Pardoux [7]). To derive (8), fix t, and note that

$$\sigma_t^V(f) = \psi_t\theta_t$$

$$\psi_t = \tilde{E}[Z_t \mid F_t] = \exp\int_0^t \langle\pi_s(h), dy_s\rangle - \tfrac{1}{2}\int_0^t |\pi_s(h)|^2 ds$$

$$\theta_t = E\left[f(x_t)\exp[-\int_0^t V(s,x_s)ds] \mid F_t\right] .$$

Now let $\ell_s = E\{f(x_t)\exp\left[-\int_0^t V(u,x_u)du \mid G_{s+}\right]$. Since x_s is a strong Markov process independent of $W(\cdot)$ (Stroock and Varadhan [10])

$$\begin{aligned}\ell_s &= \exp(-\int_0^s V(u,x_u)du)E_{sx_s}\{f(x_t)\exp -\int_s^t V(u,x_u)du\} \\ &= \exp(-\int_0^s V(u,x_u)du)(\bar{\Pi}_{st}f)(x_s) .\end{aligned}$$

We can apply a theorem of Liptser and Shiryayev [11] on equations of optimal extrapolation to conclude that for $u < s \leq t$

$$E\{\ell_s \mid F_u\} = E\ell_s + \int_0^u \left[E\{\ell_\tau h(x_\tau)|F_\tau\} - \pi_\tau(h)E\{\ell_\tau|F_\tau\}\right] \times \left[dy_\tau - \pi_\tau(h)d\tau\right] .$$

By applying Ito's rule, we then discover

$$d_u\psi_u E[\ell_t \mid F_u] = \langle \psi_u E(\ell_u h(x_u) \mid F_u), dy_u \rangle$$

$$= \langle \psi_u E\Big[h(x_u)(\bar{\bar{\Pi}}_{st} f)(x_u) \exp -\int_0^u V(\tau, x_\tau)\, d\tau \mid F_u\Big], dy_u \rangle$$

$$= \langle \sigma_u^V(h\bar{\bar{\Pi}}_{ut} f), dy_u \rangle \ .$$

Since $\psi_0 E[\ell_t \mid F_0] = E\bar{\bar{\Pi}}_{0t} f$, integration of the above implies

$$\sigma_t^V(f) = \psi_t E[\ell_t \mid F_t] = E\bar{\bar{\Pi}}_{0t} f + \int_0^t \langle \sigma_s^V(h\bar{\bar{\Pi}}_{st} f), dy_s \rangle$$

as desired.

Clearly then, to prove theorem 2 it suffices to show that $\tilde{\sigma}_t^V(f)$ satisfies (8) also. This is done by introducing the solution v of a p.d.e. that is adjoint to (7), at least insofar as concerns the deterministic part. Fix t and consider

$$\begin{aligned} &\frac{\partial v}{\partial s} + A_s v = 0 \\ &\qquad v(t,x) = f(x) \\ &\qquad v \in L^2(0,T; H^1(\mathbb{R}^N)) \ . \end{aligned} \tag{9}$$

We then have for the fixed t, $\tilde{\sigma}_t^V(f) = \langle u(t), v(t) \rangle$.

<u>Lemma 2</u>. (Feynman-Kac)

For $f \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, (9) has the unique solution

$$\begin{aligned} v(s,x) &= E_{sx} f(x_t) \exp(-\int_s^t V(u, x_u)\, du) \\ &= (\bar{\bar{\Pi}}_{st} f)(x) \ . \end{aligned}$$

<u>Proof</u>. This result is well known for sufficiently regular f, V, m, σ and h. When f, V, etc. satisfy only $A_1) - A_3)$ approximate them by regular f^n, V^n, etc. and take limits. Again, details may be inferred from analogous arguments in Pardoux [9].

We are now ready to complete the proof of theorem 2. For $s < t$, $f \in L^\infty(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$

$$d\langle u(s),v(s)\rangle = \langle A_s^* u(s),v(s)\rangle ds + \sum_{k=1}^{p} \langle h_k u(s),v(s)\rangle dy_k(s)$$
$$+ \langle u(s), - A_s v(s)\rangle ds$$
$$= \sum_{k=1}^{p} \langle u(s),h_k \bar{\Pi}_{st} f\rangle dy_k(s)$$
$$= \langle \tilde{\sigma}_s^v(h\bar{\Pi}_{st} f),dy(s)\rangle$$

and

$$\langle u(0),v(0)\rangle = \int p_0(x)\bar{\Pi}_{0t} f(x)dx = E\bar{\Pi}_{0t} f \quad .$$

Thus

$$\tilde{\sigma}_t^v(f) = \langle u(t),v(t)\rangle = E\bar{\Pi}_{0t} f + \int_0^t \langle \tilde{\sigma}_s^v(h\bar{\Pi}_{st} f),dy_s\rangle \tag{10}$$

$\forall f \in L^\infty(R^N) \cap L^2(R^N)$. By taking limits, (10) will hold for all $f \in L^\infty(R^N)$, and hence, by lemma 2, we are done.

An alternate approach to theorem 2 exists and involves a generalized Feynman-Kac rule. Instead of (9), the full adjoint to (7) is considered: for fixed t, consider

$$\begin{aligned} dv(s) + A_s v(s) + \ hv(s),dy(s) \ &= 0 \\ v(t,x) &= f(x) \\ v(\cdot) &\in L^2(\Omega \times [0,T]; H^1(R^N)) \quad . \end{aligned} \tag{11}$$

(11) must be interpreted as a backwards equation, i.e. $v(s,x)$ is adapted to $F_t^s := \sigma\{y_\tau \mid s < \tau < t\}$ since the initial condition is imposed at t. Again by modifying the techniques of Pardoux [9], we obtain a generalization of his results.

<u>Theorem 3</u>. For $f(x) \in L^2(R^N) \cap L^\infty(R^N)$. (11) has a unique solution, which, moreover may be expressed as

$$v(s,x) = \tilde{E}_{sx}\{f(x_t)\exp[-\int_s^t V(x_\tau)d\tau]Z_t^s \mid F_t^s\}$$

where $Z_t^s = \exp[\int_s^t \langle h(x_\tau),dy_\tau\rangle - \tfrac{1}{2}\int_s^t \|h(x_\tau)\|^2 d\tau]$.

Theorem 2 then follows from theorem 3, because

$$d\langle u(s),v(s)\rangle = 0$$

implying that

$$\tilde{\sigma}_t^V(f) = \langle u(t),v(t)\rangle = \langle u(0),v(0)\rangle$$

$$= \tilde{E}\{f(x_t)\exp[-\int_0^t V(x_s)ds]Z_t \mid F_t\} \quad \text{a.e.}$$

as desired.

For the purposes of application in section 3, we state yet one more variant of the Zakai equation theorem. Retain the processes x_t and y_t defined in 2.1. However, suppose now that an open domain O with C^2-boundary ∂O is given, and consider the problem of finding $u(t,x)$ such that

$$\tilde{E}\{f(x_t)e^{-M(x_0)}1_{\{t<\tau\}}\exp[-\int_0^t V(s,x_s)ds]Z_t \mid F_t\}$$

$$= \int_O dx\, u(t,x)f(x) \quad . \tag{12}$$

In (12), M is a function bounded on compacts in O and τ denotes the exit time from O; it is assumed that the initial density $p_0(x)$ satisfies $\text{supp}\, p_0(x) \subset O$. (The term $e^{-M(x_0)}$ might appear odd, but this is necessary for an application in the next section.) The appropriate Zakai equation should then be

$$\left.\begin{aligned} & du(t) = A_t^* u(t)dt + u(t)\langle h(t,x),dy_t\rangle \\ & u(0,x) = p_0(x)e^{-M(x)} \\ & u(t,\cdot) \in L^2(\Omega \times [0,T];\, H_0^1(O)) \quad . \end{aligned}\right\} \tag{13}$$

As usual, $H_0^1(O)$ denotes the completion in $H^1(O)$ of the infinitely differentiable functions with compact support in O. The proof of the next theorem is analogous in all respects to that of theorem 2.

Theorem 4. Let $e^{-M(x)}p_0(x) \varepsilon L^\infty(O) \cap L^2(O)$ and $f \varepsilon L^\infty(O)$. (13) has a unique solution $u(t,x)$ that, in addition, satisfies (12).

3. An application

3.1. Signal models with entrance boundaries

In a study of Lie algebraic techniques in filtering theory (Ocone [5,6]), we were led to models with scalar signals for which the local drift $m(x)$ satisfied

$$m'(x) + m^2(x) = V(x) \tag{14}$$

for certain functions $V(x)$. Such signals evolve only in bounded or semi-infinite intervals O, in general, and exhibit entrance boundary behavior at the (finite) endpoints of O. To see this, suppose $k(x)$ is a solution of

$$k''(x) = V(x)k(x) \quad ;$$

then $m(x) = k'(x)/k(x)$ certainly solves (14). However, if $k(r) = 0$, then $m(x)$ becomes singular at r, and, in fact $m(x) \sim \frac{1}{x-r}$ as $x \to r$. Thus the typical solution $m(x)$ of (14) will be defined on an interval $O = (r_0, r_1)$ at the (finite) endpoints of which it has simple poles. From the theory of stochastic differential equations (Gihman-Skorohod [4]), given a r.v $\eta \in O$ a.s, an O-valued process x_t exists such that

$$dx_t = m(x_t)dt + db_t, \; x_0 = \eta \quad , \tag{15}$$

and the endpoints at which m is singular are entrance boundaries of x_t. Henceforth we will assume that $V(x)$ is a continuous, bounded function and that $m(x)$, O, and x_t are described as above. As usual the observation y_t will be

$$dy_t = h(x_t)dt + dW_t \tag{16}$$

and $p_0(x)$ will denote the density of η. The assumptions on h established in section 2 are maintained here.

Previously (Ocone [5,6]) we stated, but only formally derived, Zakai equations for problems defined by (14) - (16). In this section, we aim to replace the formal calculations with a rigorous result. The work here elaborates on the theme of so-called 'gauge transformations' of the Zakai equation as discussed in Mitter [3], Brockett [1], and, implicitly, in Beneš [2]. The gauge transformations relate to a simplification of the functional integration by a Girsanov transformation to a measure w.r.t which x_t is Brownian. This is the technique that led Beneš to his discovery of new finite dimensionally computable filters, and it is the central feature of the calculation to follow.

3.2. A Zakai equation for (15) - (16)

Again define the underlying probability space as (Ω, P);

$$\Omega = C[0,t] \times C[0,t]$$
$$P = Q_\eta \times \mu \quad ,$$

where Q_η is the measure induced by the solution of (15). Now define μ^η to be the measure induced on $C[0,t]$ by the process $\eta + B_t$ for a Brownian motion B_t. The first important remark is that $Q_\eta << \mu^\eta$; indeed

$$\frac{dQ_\eta}{d\mu^\eta}(x) = 1_{\{\tau > t\}} \exp \int_0^t m(x_s)dx_s - \tfrac{1}{2}\int_0^t m^2(x_s)ds$$

$$\tau = \begin{cases} \inf\{s < t \mid x_s = t\} & \text{if} \quad \exists s < t \text{ s.t. } x_s = t \\ \infty & \text{otherwise} \end{cases}$$

(Liptser and Shiryayer [11], p. 248). The main idea is to transform measures so as to work with μ^η. Accordingly, on Ω define

$$\bar{P} = \mu^\eta \times \mu$$

and
$$\frac{d\tilde{P}}{d\bar{P}} = \exp -\int_0^t h(x_s)dW_s - \tfrac{1}{2}\int_0^t h^2(x_s)ds = Z_t \quad .$$

Then $\frac{dP}{d\tilde{P}} = \frac{dP}{d\bar{P}} \frac{d\bar{P}}{d\tilde{P}}$ and

$$E\{f(x_t) \mid F_t\} = \frac{\tilde{E}\left[f(x_t) \frac{dP}{d\tilde{P}} \mid F_t\right]}{\tilde{E}\left[\frac{dP}{d\tilde{P}} \mid F_t\right]} \quad . \tag{17}$$

We wish to represent $\tilde{E}\left[f(x_t) \frac{dP}{d\tilde{P}} \mid F_t\right]$ as in section 2.

Clearly, from what has gone above

$$\frac{dP}{d\tilde{P}} = 1_{\{\tau > t\}} \exp \int_0^t m(x_s)dx_s - ½\int_0^t m^2(x_s)ds \times \exp \int_0^t h(x_s)dy_s - ½\int_0^t h^2(x_s)ds \ . \tag{18}$$

To get to the final result, we must perform the following trick on (18). Let $r \in O$ and let $M(x) = \int_r^x m(z)dz$. As is well-known, the distribution of x_s on $(\Omega,\tilde{P})$ is the same as on (Ω,P), namely, the distribution of $\eta + B_s$. Therefore by Ito's rule

$$M(x_t) - M(x_0) = \int_0^t m(x_s)dx_s + ½\int_0^t m'(x_s)ds \quad .$$

Thus, substituting $\int_0^t m(x_s)dx_s = M(x_t) - M(x_0) - ½\int_0^t m'(x_s)ds$ into (18),

$$\frac{dP}{d\tilde{P}} = e^{M(x_t)} 1_{\{\tau > t\}} e^{-M(x_0)} \exp\left[-½\int_0^t V(x_s)ds\right] Z_t \quad .$$

With this, we may write

$$\tilde{E}\{f(x_t) \frac{dP}{d\tilde{P}} \mid F_t\} = \tilde{E}\{f(x_t) e^{M(x_t)} 1_{\{\tau > t\}} e^{-M(x_0) - ½\int_0^t V(x_s)ds} Z_t \mid F_t\} \quad . \tag{19}$$

__Theorem 5.__ Define $A^* : H_0^1(O) \to H_0^1(O)$

$$\langle A^* u, v\rangle = -\tfrac{1}{2}\int_O \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\,dx - \tfrac{1}{2}\int_O V(x)u(x)v(x)\,dx \quad .$$

Assume $p_0(x)e^{-M(x)} \in L^\infty(O) \cap L^2(O)$, $f(x)e^{M(x)} \in L^\infty(O)$, and let $u(t,x)$ be the unique solution of

$$du_t = A^* u_t\, dt + u_t\, h(x)dy_t$$

$$u_0(x) = p_0(x)e^{-M(x)}$$

$$u(t) \in L^2(\Omega \times [0,T]\ ;\ H_0^1(O)) \quad .$$

Then

$$\tilde{E}\{f(x_t)\frac{dP}{d\tilde{P}} \mid F_t\} = \int_O f(x)e^{M(x)}u(x,t)dt \quad \text{a.s.} \quad \forall t \in [0,T] \quad .$$

__Proof.__ On $(\Omega,\tilde{P})$ x_s and y_s, $s < t$, are independent and x_s has the law of $\eta + B_s$. Clearly we can interpret (19) as if x_s were defined $\forall s \in (0,\infty)$ and τ = first exit time from O. Since $\tfrac{1}{2}\frac{\partial^2}{\partial x^2}$ is the generator of x_s, the theorem then follows immediately from theorem 4.

__Remark__ $p(x,t) = e^{M(x)}u(x,t)$ is, in effect, an unnormalized conditional density, and the factor $e^{M(x)}$ is the 'gauge transformation' mentioned above.

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Pardoux's results on the Zakai equation for nonlinear filtering are extended to cover the case of estimating a signal modified by a potential term. This is applied to state a rigorous Zakai equation for certain filtering problems involving signals with entrance boundaries.